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EQUILIBRIUM TRAFFIC ASSIGNMENT FOR LARGE SCALE TRANSIT NETWORKS

by

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ABSTRACT

This paper investigates the equilibrium traffic assignment methodology for large scale transit networks. Variational inequality formulations for the equilibrium traffic assignment are developped using a particular graph structure called a hyperpath. Computational techniques for shortest hyperpaths, as well as algorithms for computing equilibrium flows are discussed.

Keywords : traffic assignment, transit network, shortest paths, variational inequality, equilibrium flow.

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RÉSUMÉ

Nous étudions dans cet article une méthodologie d'affectation des passagers sur un réseau de transport public de grande taille. A l'aide d'un sous-graphe particulier, appelé ici hyperchemin, nous formulons le problème d'affectation de trafic d'équilibre comme un problème d'inégalité variationnelle. Nous analysons finalement les techniques de calcul des plus courts hyperchemins et des flots d'équilibre.

1. INTRODUCTION

A major obstacle in developing a passenger routing model for transit networks stems from the difficulty in modelling the passenger route choice behavior at a transit stop shared by several competitive transit lines.

Consider a passenger waiting at a stop A shared by several bus lines which can transport this passenger to a given destination B. The decision problem faced by the passenger, in its simplest form, is whether to board an arriving bus or to wait for a faster one.

From the traffic assignment point of view, the problem reduces to determining the subset of attractive lines at stop A, such that the passenger always boards the first bus on these lines to reach destination B.

The difference between existing traffic assignment models stems mainly from the way the subset of attractive transit lines is selected at each stop. For example, Lampkin and Saalmans (1967), Andreasson (1976) and Hasselströem (1981) proposed local heuristic rules for selecting the attractive transit lines. Chriqui and Robillard (1975) appear to be the first to define explicitly the attractive set between two consecutive points as a subset of transit lines which minimizes the rider's expected travel time – the sum of the mean wait and in-vehicle times. Under the simplifying assumptions that passengers arrive randomly at the stop node, and all bus routes are statistically independent with given exponential headway distribution, Chriqui and Robillard showed that the optimal subset of transit lines which minimizes the passenger's expected travel time is a solution to a Boolean hyperbolic program, which may be solved efficiently with a greedy type algorithm (Robillard, 1971).

Recently, continuing in the same vein as the work performed by Chriqui and Robillard, Marguier (1981), Marguier and Ceder (1984) analyzed the optimal strategies for selecting an attractive set of transit lines at a stop, by taking into account the amount of time the passenger has already waited when the first bus arrives. In a different direction, Spiess (1983, 1984) developed an optimal route choice strategy, and Gendreau (1984) presented a detailed theoretical study of an equilibrium traffic assignment model for transit networks. In broad terms, a general strategy is a set of coherent decisional rules that, when applied, allows the passenger to travel from his origin to his destination (Spiess, 1983; Gendreau, 1984). However, for practical purpose, only strategies, which are obtained by specifying, for each node of a subset of nodes, a set of attractive transit lines together with their correspopnding transfer nodes, may be considered. An optimal strategy is, therefore, a strategy which minimizes the passenger's travel time. Under the simplifying assumptions mentioned earlier, Spiess formulated an optimal strategy traffic assignment model as a linear program, when the in-vehicle travel times are flow independent constants, and as a convex program with linear constraints, when the in-vehicle travel times are convex functions. The proposed solution procedure for the linear case has the flavor of a shortest path algorithm.

This paper investigates the equilibrium traffic assignment methodology for large scale transit networks. Underlying the traffic assignment models considered in this research are the local optimization model developed by Chriqui and Robillard and the strategy concept introduced by Spiess.

The objective is to provide a better understanding of the network equilibrium methodology for large scale transit networks. This is achieved, in part, by focusing clearly on the underlying graph structure of a strategy.

It will be shown that, by explicitly defining the flow and cost structures of this graph – called here a hyperpath to stress its relationship with the usual path concept – standard equilibrium traffic assignment model and formulation may now be developed for transit networks simply by substituting the hyperpath flow space for the path flow space of existing models for road networks. Such modelling flexibility has not been obtained so far. Also, from a computational standpoint, the intimate relationship between a path and a hyperpath enables the adaptation of the most advanced techniques for the computation of shortest paths and equilibrium flows in road networks, to that of hyperpaths and flows in transit networks.

The next section defines a generic transit network, introduces the notation used, formally defines the notion of hyperpath and discusses the resulting flow and cost structures. Section three develops various compact formulations of the traffic equilibrium model and discusses potential algorithms for computing an equilibrium flow. The last section analyzes the determination of shortest hyperpaths which is the essence of the column generating scheme implicit in any mathematical model formulated in the hyperpath flow space. Proofs of theorems are collected in the appendix.

2. TRANSIT NETWORKS AND HYPERPATHS

Consider an urban transit network consisting of a set of distinct transit lines, transit stops and walking links. This network is abstracted into a graph model, in which a typical stop node served by several transit lines is represented by a subgraph in which each transit line is identified by a distinct boarding arc. Such a subgraph is illustrated in figure 1.

In order to integrate the pedestrian and transit networks, a pedestrian path is implicitly considered as a transit line with appropriate

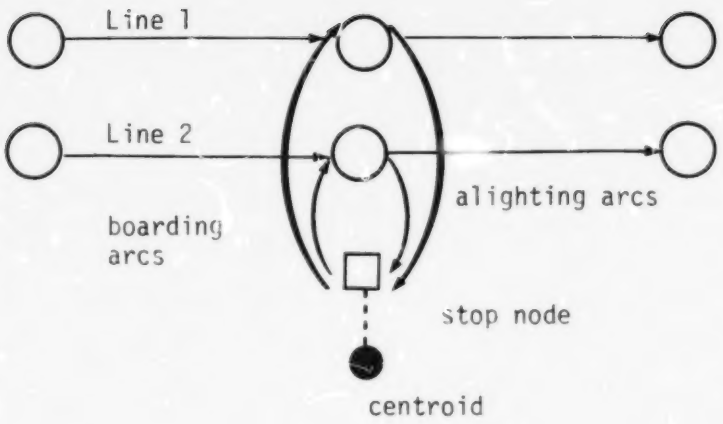


Figure 1 : Graphical representation of a transit stop served by several lines

characteristics. The above detailed graphical structure is aimed at facilitating the description of hyperpaths; a more compact representation of the pedestrian-transit network may be used in practice.

Throughout the paper, the following general notation will be used :

$G = (N,E)$ = graph modeling the transit-pedestrian network;

$N = (i \mid i=1,2,\ldots,n)$ = node set;

E = arc set;

N^B = set of stop nodes;

E^B = set of boarding arcs, $E^B \subset E$;

x_{ij} = aggregated flow on arc $(ij) \in E$;

$X = (x_{ij})$ = vector of aggregated flows;

$C_{ij}(X)$ = vector function with components $C_{ij}(X)$;

Φ_{ij} = mean frequency of the transit line associated with the boarding arc $(i,j) \in E^B$;

$N_i^+ = (j \mid (i,j) \in E)$; $N_i^- = (j \mid (j,i) \in E)$.

An index p will be added to the above variables when they are related to a particular hyperpath p.

Depending on the arc type, the travel cost C_{ij}, may be a constant or a function $C_{ij}(X)$ of the flow X. C_{ij} may be considered as a generalized cost combining different factors such as travel time, fare, comfort and the residual capacity of the carrier. The specification and calibration of such functions are a research topic in its own right and will not be discussed here. Specific assumptions of the cost C_{ij} will be stated as needed. It is assumed that the origin-destination flows are known and fixed as are the transit line frequencies. Frequencies of the pedestrian paths are fixed at a common arbitrary large value.

Central to this paper is the concept of a hyperpath (h-path) connecting two points. For clarity, a single origin-destination pair (s,t) is first considered.

Definition 1 - A subgraph $G_p = (N_p, E_p)$ is a hyperpath connecting s and t if it satisfies :

1) G_p is acyclic,

2) $s \in N_p$, $t \in N_p$ and $N^-_{sp} = N^+_{tp} = \emptyset$,

3) $N^+_{ip} \neq \emptyset$ and $N^-_{ip} \neq \emptyset$, for all $i \in N_p - \{s,t\}$.

Definition 2 - A vector $\pi_p = (\pi_{ijp})$ is a characteristic vector for hyperpath p if it satisfies :

1) $\sum_{j \in N^+_i} \pi_{ijp} = 1$, for all $i \in N_p$,

2) $\pi_{ijp} > 0$, for all $(i,j) \in E$,

3) $\pi_{ijp} = 0$, for all $(i,j) \notin E_p$.

For the transit network considered, and under the assumptions of random passenger arrivals at the bus stops, and exponential distribution of transit line headways with parapeters $1/\phi_{ij}$, we restrict our analysis to a subclass of h-paths for which the characteristic vector π_p is uniquely defined as follows :

$$\pi_{ijp} = \begin{cases} \phi_{ij} \Big/ \sum_{\ell \in N^+_{ip}} \phi_{i\ell}, & \text{for all } (i,j) \in N^B_p \quad (1) \\ 1, & \text{for all } (i,j) \in E_p - E^B_p \quad (2) \\ 0, & \text{otherwise.} \quad (3) \end{cases}$$

Underlying this definition is the rule that passengers board the carriers of the attractive set in proportion to their frequencies and that the two alternatives of either staying on the same carrier or alighting this carrier to board another one are mutually exclusive.

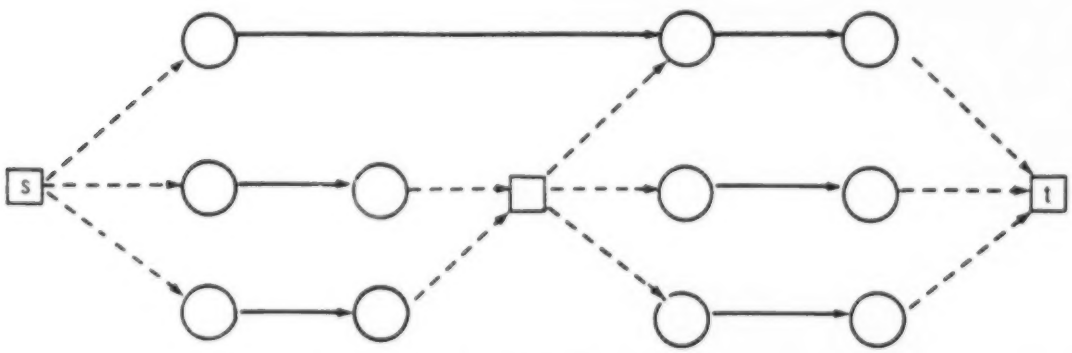


Figure 2 : A h-path connecting s and t

Let Q_p denote the set of all paths of G_p connecting s and t. To each individual path $q \in Q_p$ is associated a unique weight ω_{qp} :

$$\omega_{pq} = \prod_{(i,j) \in E_p} \pi_{ijp}^{\delta_{ijq}}, \qquad q \in Q_p, \quad (4)$$

where δ_{ijq} = 1 arc (i,j) belongs to path q and 0 otherwise.

These weights satisfy

Theorem 1 - $$\sum_{q \in Q_p} \omega_{qp} = 1 , \quad (5)$$

and represent the utilization rates of the paths contained in h-path p.

Let y_p denote the flow on h-path p. The resulting path flows y_{qp} are then

$$y_{qp} = \omega_{qp} y_p , \qquad q \in Q_p , \quad (6)$$

and induce the arc flows x_{ijp} :

$$x_{ijp} = \sum_{q \in Q_p} \pi_{ijp}\, y_{qp}, \qquad (i,j) \in E \tag{7}$$

The aggregated arc flows x_{ij} are obtained by summing the h-path arc flows x_{ijp} over all h-paths of all origin-destination pairs :

$$x_{ij} = \sum_{p} x_{ijp}. \tag{8}$$

At first glance, the h-path framework is highly impractical, since computing the arc flows x_{ijp} seems to require an exhaustive enumeration of paths of Q_p. Fortunately, the arc flows can be computed sequentially in a topological order which obviates path enumeration as shown next.

Let β_{ip} denote the utilization rate of node i

$$\beta_{ip} = \sum_{q \in Q_p} \delta_{iq}\, \omega_{qp}, \qquad i \in N_p, \tag{9}$$

and α_{ijp} that of arc (i,j) :

$$\alpha_{ijp} = \sum_{q \in Q_p} \delta_{ijq}\, \omega_{qp}, \qquad (i,j) \in E \tag{10}$$

where $\delta_{iq} = 1$ if q passes through node i; 0 otherwise.

Clearly

$$\beta_{sp} = \beta_{tp} = 1 \tag{11}$$

and from equations (6) and (7) we have

$$x_{ijp} = \alpha_{ijp}\, y_p. \tag{12}$$

The following results provide an efficient sequential procedure for the computation of the arc utilization rates; and thus, the arc flows via equation (12) :

Theorem 2 - $$\beta_{jp} = \sum_{i \in N_j^-} \beta_{ip}\, \pi_{ijp}, \qquad j \in N_p. \tag{13}$$

Theorem 3 - $$\alpha_{ijp} = \beta_{ip}\, \pi_{ijp}, \qquad (i,j) \in E_p. \tag{14}$$

Similar results have been obtained by Dial (1971) and Robillard (1973) for the road network assignment problem. Figure 3 illustrates the two ways of distributing h-path flow y_p into arc flows x_{ijp}.

We now examine the cost attributes of an h-path. In addition to the usual arc cost C_{ij}, a fixed unit waiting cost w_{ijp} is asociated with each arc :

$$w_{ijp} = \begin{cases} (b/\phi_{ij})\pi_{ij}, & (i,j)\in E_p^B, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

where b is a positive weight exogenously chosen. This definition is a direct consequence of the assumptions invoqued earlier. Note that the waiting costs vary from one h-path to another. Assuming the standard additive cost structure, the cost of any path $q\in Q_p$ is simply the sum of the costs of the arcs forming q :

$$f_{qp} = \sum_{(i,j)\in E} \delta_{ijqp} (C_{ij}+w_{ijp}). \tag{16}$$

Finally, the cost of h-path p is

$$f_p = \sum_{q\in Q_p} \omega_{qp} f_{qp}, \tag{17}$$

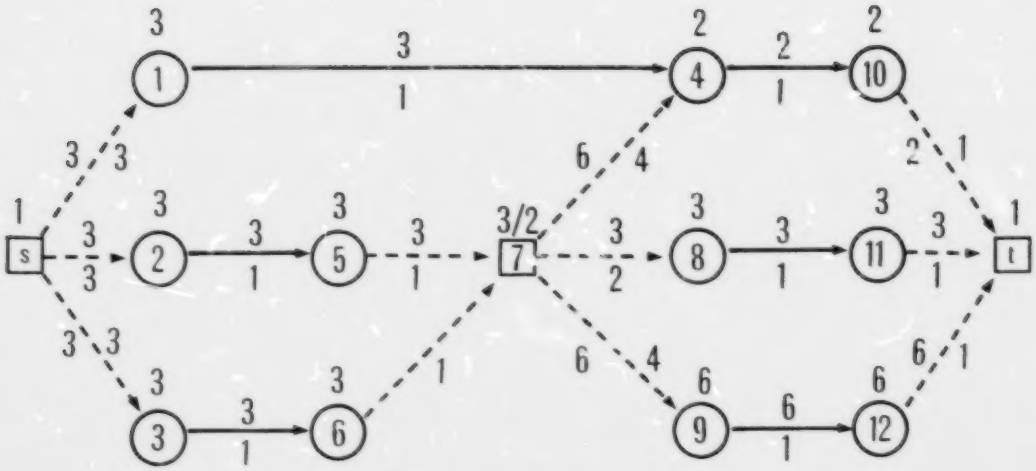
or equivalently

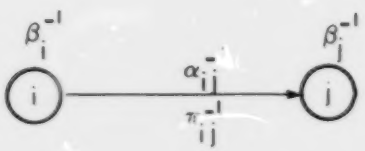
$$f_p = \sum_{(i,j)\in E} \alpha_{ijp} (C_{ij}+w_{ijp}). \tag{18}$$

3. DETERMINING THE ACTIVE HYPERPATHS : AN APPLICATION OF WARDROP'S FIRST PRINCIPLE

Consider a transit demand $g = (g_k)$, where g_k is the demand associated with the k^{th} origin-destination pair. The demand g_k subdivides into h-path flows y_p^k, $p\in P_k$. Let $Y = (y_p^k)$.

If we assume that all transit passengers travel on shortest h-paths, then any resulting equilibrium h-path flow Y* must satisfy :





q	Path q	ω_q	y_q
1	(s,1,4,10,t)	1/3	40
2	(s,2,5,7,4,10,t)	1/12	10
3	(s,2,5,7,8,11,t)	1/6	20
4	(s,2.5.7.9,12,t)	1/12	10
5	(s,3,6,7,4,10,t)	1/12	10
6	(s,3,6,7,8,11,t)	1/6	20
7	(s,3,6,7,9,12,t)	1/12	10
		1	120

(i,j)	$x_{ij} = \alpha_{ij} \cdot y$
(1,4)	40
(2,5)	40
(3,6)	40
(4,10)	60
(8,11)	40
(9,12)	20

Figure 3 : Hyperpath aggregated node and arc weights

Conservation of flow equations

$$\sum_{p \in P_k} y_p^{k*} = g_k, \qquad \text{for all O/D pair } k \tag{19}$$

$$Y^* \geq 0; \tag{20}$$

Wardrop conditions (Wardrop, 1952)

$$f_p^k(Y^*) = h_k, \qquad \text{if } y_p^k > 0, \tag{21}$$

and

$$h_k = \min_{p \in P_k} \{ f_p^k(Y^*) \}. \tag{22}$$

Note that any feasible Y induces a unique aggregated arc flow X via equation (12). As shown by Smith (1979), Y^* is a solution to the following variational inequality problem (VIP) :

$$F(Y^*)(Y-Y^*) \geq 0, \qquad \text{for all } y \in \Omega \tag{23}$$

in which F(Y) is the vector $(f_p^k(Y))$, Ω is the polyhedral set of feasible h-path flows defined by equations (19) and (20).

Assume now that C(X) is a positive continuous strictly monotone function, then equation (18) implies that F(Y) is also a positive continuous function of Y. This ensures the existence of a solution Y^* which is not necessarily unique. (See for example Smith, 1979; Dafermos, 1980).

Noting that problem (23) can be rewritten as

$$C(X^*)(X-X^*) + W(Y-Y^*) \geq 0, \qquad \text{for all } Y \in \Omega, \tag{24}$$

in which W is the waiting cost vector with components

$$w_p^k = \sum_{(i,j) \in E} w_{ijp}^k \alpha_{ijp}^k, \qquad p \in P_k; \quad \text{all } k, \tag{25}$$

it is easy to show that

Theorem 4 - The equilibrium aggregated arc flow X^* and the total waiting cost WY^* are unique.

From a computational point of view, formulation (24) provides crucial information concerning the computer memory space required for the implementation of solution algorithms for transit network problems. Effectively, to use inequality (24), only the aggregated arc flows X and the values U=W.Y of the total waiting costs are needed, and not the explicit hyperpath flows Y.

For example, consider the following basic feasible direction algorithm for problem (24).

Basic improving feasible direction algorithm

Step 1 - Determine a feasible aggregated arc flow X^1. Let U^1 denote the associated total waiting cost. Set r=1.

Step 2 - If

$$C(X^r)(Z^s-X^r) + V^s-U^r \geq 0, \quad \text{for all } s \tag{26}$$

(where Z^s and V^s are the aggregated flow and the total waiting cost correspopnding to the s^{th} extremal hyperpath flow respectively)

terminate : $X^* = X^r$;

otherwise go to step 3.

Step 3 - Determine a scalar $0 \leq \theta^* \leq 1$ satisfying

$$[C(X^r+\theta^*(\tilde{X}-X^r))(\tilde{X}-X^r) + \tilde{U}-\tilde{U}](\theta-\theta^*) \geq 0$$

for every $0 \leq \theta \leq 1$, (27)

in which $(\tilde{X}-X^r)$ is an improving direction and $\tilde{U}$ is the total waiting cost associated with $\tilde{X}$.

Step 4 - Revise the current iteration

$$X^{r+1} = X^r+\theta^*(\tilde{X}-X^r), \tag{28}$$

$$U^{r+1} = U^r+\theta^*(\tilde{U}-U^r), \tag{29}$$

$$r = r+1.$$

Return to step 2.

Note first that step 2 reduces to the computation of shortest hyperpaths, which will be shown in the next section, to require only marginally more effort and memory space than the computation of shortest paths. Naturally, specific implementations of the above procedure will depend on the particular choice of the improving direction. Possible alternatives include Frank and Wolfe (1956)'s extremal descent direction, Nguyen and Dupuis (1984)'s improving direction, or Marcotte (1984)'s convex combination of extremal directions.

Regardless of the specific choice for the improving direction, it is clear that the memory space requirement of the above basic framework is of the order of the number of arcs of the transit network, and not of the number of hyperpaths. More specifically, from an algorithmic point of view, each feasible solution is completely identified by its vector of aggregated arc flow and the corresponding value of the total waiting cost. This crucial result has not been noted so far.

4. COMPUTING SHORTEST HYPERPATHS

For the purpose of clarity, consider a single destination t. Let p(i) denote an h-path from node i to t. We will first establish a recursive formula for the cost of p(i), which is essential for the computation of shortest h-paths.

Definition 3 - An h-path $p^i(j)$ is a sub-h-path of p(i), if

$$\pi_{uvp^i(j)} = \pi_{uvp(i)}, \qquad \text{for all } (u,v)\in E_{p^i(j)}.$$

Define $f_{p(i)}=0$, if $N_{p(i)}=\{i\}$. Also, without loss of generality, assume $C_{ij}=0$ for every boarding arc $(i,j)\in E^B$.

Theorem 5 - The travel cost of any h-path p(i) can be expressed as

$$f_{p(i)} = \sum_{j\in N^+_{ip(i)}} \pi_{ijp(i)}\,(f_{p^i(j)} + w_{ijp(i)} + C_{ij}). \qquad (30)$$

Note that equation (30) may be rewritten as

$$f_{p(i)} = \begin{cases} f_{p^i(j)} + c_{ij}, & i \in N^B \\ (\sum_{j \in N^+_{ip(i)}} \phi_{ij} f_{p^i(j)} + b) / \sum_{j \in N^+_{ip(j)}} \phi_{ij}, & i \in N^B \end{cases}$$

by using definitions (1)-(3) and (15).

Consider two nodes j and $k \in N_i^+$. We say that p(j) and p(k) are mergeable if

$$N^+_{up(j)} = N^+_{up(k)}, \qquad \text{for every } u \in N_{p(j)} \cap N_{p(k)}. \tag{31}$$

In other words, if p(j) and p(k) are not mergeable, they cannot be sub-h-paths of the same h-path p(i). If p(j) and p(k) are the shortest h-paths, then either they are mergeable or it is always possible to construct an h-path p'(j) with $f_{p'(j)} = f_{p(j)}$ which is mergeable with p(k). Thus, without loss of generality, it will be assumed that the shortest h-paths p(j) and p(k) always satisfy property (31).

Let $S = (S_i)$, where each component S_i represents the length of a shortest h-path connecting i and destination t. The following equations are analogous to Bellman's equations and admit similar proofs.

Theorem 6 - S is a unique solution to

$$S_t = 0,$$

$$S_i = \begin{cases} \min_{T_i \subseteq N_i^+} \{(\sum_{i \in T_i} \phi_{ij} S_j + b) / \sum_{j \in T_i} \phi_{ij})\}, & \text{if } i \in N^B, \\ \min_{j \in N_i^+} \{S_j + C_{ij}\}, & \text{otherwise.} \end{cases} \tag{32}$$

These equations can be solved with the following iterative procedure.

Basic shortest h-paths computational algorithms

Step 1 (Initialization) - $S_t = 0$; $S_i = +\infty$, $i \neq t$, $i \in N$.

Step 2 (Updating the node labels)

2.1 If a node $i \in N^B$ exists such that $S_i > S_j + C_{ij}$, set $S_i = S_j + C_{ij}$ and return to step 2;

2.2 If a node $i \in N^B$ exists such that :

$$S_i > S_i^* = \min_{T_i \subseteq N_i^+} \{ (\sum_{j \in T_i} \phi_{ij} S_j + b) / \sum_{j \in T_i} \phi_{ij} \},$$

set $S_i = S_i^*$ and return to step 2;

otherwise terminate : the S_i are shortest h-path lengths.

Following the standard line of proof used in the classical shortest path problem, we can show that this procedure is finite and solves the shortest h-path problem. However, from a practical point of view, the above procedure is feasible only if the combinatorial subproblem embedded in step (2.2) can be solved efficiently. In fact, step (2.2) can be solved efficiently with a greedy type algorithm based on the following characterization of the obtimum set T_i^* :

Theorem 7 - $S_i = (\sum_{j \in T_i^*} \phi_{ij} S_j + b) / \sum_{j \in T_i^*} \phi_{ij}$, if and only if

$$S_i \leq S_j, \qquad j \in T_i^*, \tag{34}$$

and

$$S_j \leq S_i, \qquad j \in T_i^*. \tag{35}$$

Thus, to find the optimum set T_i^* we may apply the following algorithm (also suggested in Chriqui and Robillard (1975) in a slightly different form).

Algorithm for determining T^*_i

Step 1 - Sort the nodes $j \in N_i^+$ in increasing order of S_j, such that $S_{j1} \leq S_{j2} \leq S_{jn}$; where $n = |N_i^+|$;
set $T_i^* = \{j1\}$, $S_i = S_{j1} + b/\phi_{ij1}$ and $\Phi = \phi_{ij1}$; set $r = 2$.

Step 2 - (Updating T_i^*)
If $r > n$ or $S_{jr} > S_i^*$, terminate;
otherwise set

$$\Phi = \Phi + \phi_{ijr};\ S_i^* = S_i^* - (S_i^* - S_{jr})\ \phi_{ijr}/\Phi;$$

$$T_i^* = T_i^* \cup \{jr\};\ r = r + 1;$$

return to step 2.

A slightly more complicated algorithm will result if the nodes $j \in N_i^+$ are not sorted in increasing order of S_j.

Various labeling procedures can be implemented to carry out the shortest h-path computations. Essentially, a labeling procedure would run as follows. Every node i which receives an improved label S_i is placed in a list of candidates, say list L. Initially, only node t is in L. In a general step, a particular node i is removed from L, and the label of every node j of the set N_i^- is checked and corrected when necessary. The procedure terminates when list L is empty.

This general labeling scheme may be either a label-correcting or a label-setting algorithm when all the costs are nonnegative. In the latter case the candidate nodes of list L are processed in order of increasing label-values (an algorithm of this type was embedded in Spiess (1983, 1984) traffic assignment procedure); any other processing order yields a label-correcting type algorithm.

The limited experimental results obtained so far in shortest h-path computations show that a label-correcting algorithm performs better than a label-setting algorithm. A detailed computational study of the labeling algorithms, including time and space complexity analyses is reported in Nguyen and Pallottino (1985).

CONCLUSION

This paper provides new insight into the traffic equilibrium modeling approach for transit networks, and suggests a practical methodology for solving large-scale traffic assignment problems with known traffic demand.

The new hyperpath framework provides two main advantages. First, many computational techniques in network theory can readily be generalized to handle hyperpaths. In particular, it is shown that efficient computational procedures for shortest hyperpaths can be derived from the existing shortest path methodology.

Secondly, from a mathematical point of view, the traffic equilibrium model, formulated as a variational inequality problem in the hyperpath flow space, is identical with the one obtained for the road network in the path flow space. This matching permits the application to transit networks of the latest theoretical and algorithmic developments achieved in the network equilibrium field.

In spite of the limitations inherent in the deterministic traffic equilibrium framework, the suggested methodology seems to strike a balance between an ideal behavioral modeling and the model's computational tractability, and is therefore well suited for large-scale applications.

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Appendix A

We will first introduce some additional notation. Consider a hyperpath $G_p = (N_p, E_p)$ connecting s an t. If path q goes through node i, then let q'(i) be the portion from s to t and q(i) the portion from i to t. We write

$$q = q'(i) \cup q(i), \qquad \omega_{qp} = \omega_{q'(i)} \cdot \omega_{q(i)},$$

and define $\omega_{q'(s)} = \omega_{q(t)} = 1$. Note that $q(s) = q'(t) = q$.

Theorem 1 - $$\sum_{q \in Q_p} \omega_{qp} = 1. \tag{A.1}$$

Proof - We will prove, by induction, that

$$\sum_{q(i)} \omega_{q(i)} = 1, \quad \text{for any } i \in N_p. \tag{A.2}$$

(A.1) then follows by letting i=s. Let r_i be the number of arcs of a subpath q(i) having the maximum number of arcs. If we define $L_m = \{i \mid r_i = m\}$, then clearly $\sum_{q(i)} \omega_{q(i)} = \pi_{itp} = 1$, for all $i \in L_1$.

Suppose that (A.2) is true for all $i \in L_u$, $u=1,2,\ldots,k-1$; we show that it is true for all $i \in L_k$. Consider any $i \in L_k$, if $j \in N^+_{ip}$ then $j \in L_u$, for $u < k$. Il follows that

$$\sum_{q(i)} \omega_{q(i)} = \sum_{j \in N^+_i} \pi_{ijp} \sum_{q(j)} \omega_{q(j)}, = \sum_{j \in N^+_{ip}} \pi_{ijp} = 1, \quad i \in L_k,$$

which proves (A.2).

Theorem 2 - $$\beta_{jp} = \sum_{i \in N^-_{jp}} \beta_{ip} \pi_{ijp}, \quad j \in N_p$$

Proof - Recall that $\beta_{jp} = \sum_q \delta_{jp} \omega_{qp}$, which can be rewritten as

$$\beta_{jp} = \sum_{q'(j)} \omega_{q'(j)} \cdot \sum_{q(j)} \omega_{q(j)} = \sum_{i \in N^-_{jp}} \pi_{ijp} \sum_{q'(i)} \omega_{q'(i)}. \tag{A.3}$$

Replacing $\sum_{q'(i)} \omega_{q'(i)}$ by β_{ip}, we obtain $\beta_{jp} = \sum_{i \in N_{jp}^-} \pi_{ijp} \beta_{ip}$, $j \in N_p$.

Theorem 3 - $\alpha_{ijp} = \beta_{ip} \pi_{ijp}$, $(i,j) \in E_p$ (A.4)

Proof - Recall that $\alpha_{ijp} = \sum_q \delta_{ijq} \omega_{qp}$, which can be rewritten as

$$\alpha_{ijp} = \sum_{q'(i)U(i,j)} \omega_{q'(i)U(i,j)} \sum_{q(j)} \omega_{q(j)} = \pi_{ijp} \sum_{q'(i)} \omega_{q'(i)};$$

which proves (A.4) considering (A.3).

Theorem 5 - $f_{p(i)} = \sum_{j \in N_{ip(i)}^+} \pi_{ijp(i)} (f_{p\,i(j)} + w_{ijp(i)} + C_{ij})$ (A.5)

Proof - For the purpose of clarity, the p(i) will be temporarily dropped.

We distinguish two cases :

a) $i \notin N^B$; $w_{ij} = 0$. Clearly $f = C_{ij} + f_{p\,i(j)}$, which verifies (A.5)

b) $i \in N^B$; $C_{ij} = 0$, $j \in N_i^+$. First note that $w_{ij} = w_{ik}$, j and $k \in N_i^+$.

Define $w_i = w_{ij}$, for any $j \in N_i^+$, and rewrite (A.5) as

$$f = w_i + \sum_{j \in N_i^+} \pi_{ij} f_{p\,i(j)}. \quad (A.6)$$

Now, recall that $f = \sum_{q \in Q} \omega_q f_q$, which can be rewritten as

$$f = \sum_{j \in N_{ip(i)}^+} \pi_{ij} \sum_{q(j)} \omega_{q(j)} (w_{ij} + f_{q(j)}),$$

since $q = (i,j) \cup q(j)$, $j \in N_i^+$.

Therefore $f = w_i + \sum_{j \in N_i^+} \pi_{ij} \sum_{q(j)} \omega_{q(j)} f_{q(j)}$

which yields (A.6).

Theorem 6 - S_i, $i \in N$ are the lengths of the shortest h-paths connecting i and t if and only if

$$S_i = \begin{cases} \min\limits_{T \subseteq N_i^+} \{(\sum\limits_{j \in T_i} \phi_{ij} S_j + b) / \sum\limits_{j \in T_i} \phi_{ij}, & \text{if } i \in N^B, \quad (A.7) \\ \min\limits_{j \in N_i^+} \{S_j + C_{ij}\}, & \text{othewise;} \quad (A.8) \end{cases}$$

$$S_t = 0.$$

Proof - First, note the equivalency between (A.6) and

$$f_{p(i)} = (\sum_{j \in N_i^+} \phi_{ij} f_{p(i)} + b) / \sum_{j \in N_{ip(i)}^+} \phi_{ij}, \qquad \text{for } i \in N^B. \quad (A.9)$$

If p(i) is a shortest h-path, then clearly each sub h-path p(j), $j \in N_{ip(i)}^+$, must also be a shortest h-path. Furthermore, $N_{ip(i)}^+$ must be the set which satisfies

$$f_{p(i)} = \min_{p'(i)} \{f_{p'(i)}\} = \begin{cases} \min\limits_{T \subseteq N_i^+} \{(\sum\limits_{j \in T_i} \phi_{ij} S_j + b) / \sum\limits_{j \in T_i} \phi_{ij}\}, & \text{if } i \in N^B, \\ \min\limits_{j \in N_i^+} \{S_j + C_{ij}\}, & \text{otherwise.} \end{cases}$$

Conversely, by induction, it is easy to show that S_i is the length of an h-path p(i). Let S'_i be the length of a shortest h-path p'(i). Suppose that there is at least an i such that

$$S'_i < S_i, \quad i \in N. \quad (A.10)$$

Let i be the node which satisfies

$$S'_i = \min_j \{S'_j | S'_j < S_j\}.$$

Note that $i \neq t$. Without loss of generality, assume that $S'_j \; S'_i$, $j \in N^+_{ip(i)}$, then,

$$S'_i = \begin{cases} \left(\sum_{j \in N^+_{ip'(i)}} \Phi_{ij} S'_j + b \right) / \sum_{j \in N^+_{ip'(i)}} \Phi_{ij}, & i \in N^B, \\ S'_j + C_{ij}, & \text{otherwise.} \end{cases}$$

But since $S'_j = S_j$, for $j \in N^+_{ip(i)}$, it follows that $S'_i > S_i$ contradicting assumption (A.10).

Theorem 7 - $$S_i = \left(\sum_{j \in T^*_i} \Phi_{ij} S_j + b \right) / \sum_{j \in T^*_i} \Phi_i, \tag{A.11}$$

if and only if

$$S_j \geq S_i, \quad j \notin T^*_i; \tag{A.12}$$

$$S_j < S_i, \quad j \in T^*_i. \tag{A.13}$$

Proof - Let T^*_i be the optimal set. Suppose there exists a $k \in T^*_i$, such that $S_k < S_i$.

Let

$$S_i(T_i) = \left(\sum_{j \in T_i} \Phi_{ij} S_j + b \right) / \sum_{j \in T_i} \Phi_{ij},$$

$$T^+_i = T^*_i \cup \{k\}, \qquad \Phi(T_i) = \sum_{j \in T_i} \Phi_{ij};$$

we have $\Phi(T^+_i) \; S_i(T^+_i) = \Phi(T^*_i) \; S_i(T^*_i) + \Phi_{ik} \; S_k$, replacing S_k by $S_i(T^*_i)$ yields $\Phi(T^+_i) \; S_i(T^+_i) < \Phi(T^+_i) \; S_i(T^*_i)$, thus $S_i(T^+_i) < S_i(T^*_i)$, which contradicts the optimality of $S_i(T^*_i)$.

Suppose there exists a $k \in T^*_i$, such that $S_k > S_i$. Let $T^-_i = T^*_i - \{k\}$, then

$$\Phi(T^-_i) \; S_i(T^-_i) = \Phi(T^*_i) \; S_i(T^*_i) - \Phi_{ik} \; S_k < \Phi(T^-_i) \; S_i(T^*_i),$$

which contradicts the optimality of S_i.

Conversely, suppose that T_i^* satisfies (A.12) and (A.13); and let $\tilde{T}_i \neq T_i^*$ be an optimal set. Note that $\tilde{T}_i$ also satisfies (A.12) and (A.13). If there is a $k \varepsilon \tilde{T}_i$ and $k \varepsilon T_i^*$ then $S_i(T_i^*) < (S_i(\tilde{T}_i)$, which implies that T_i^* is also an optimum set.

Now suppose that $\tilde{T}_i \subset T_i^*$, and let $k \varepsilon T_i^*$ and $k \varepsilon \tilde{T}_i$. Define $T_i^- = T_i^* - \{k\}$, then

$$S_i(T_i^-) = S_i(T_i^*) + \Phi_{ik}\left(S_i(T^*_i) - S_k\right) / \Phi(T_i^-),$$

and since $S_k < S_i(T_i^*)$, we obtain $S_i(T_i^-) > S_i(T_i^*)$.

Thus, by repeating the same operation if $T_i^- \neq \tilde{T}_i$, we will finally reach the state where $T_i^- = \tilde{T}_i$. This proves that T_i^* is also an optimum set.